Relaxations of the 3-Partition Problem

Why Children's Puzzles are Hard

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Discrete Mathematics and Mathematical Programming

Problems I aim to solve

- How should these presentations be scheduled as conveniently as possible?
- What should I pack for the Christmas holiday?
- On what shall I spend my money?

How to solve these problems in general? Once you have an idea, you will need to know what can go wrong.

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Outline

The 3-partition problem
- Definition

When is it (un)solvable?
- A relaxation

When do we not know?
- Nearly-feasible instances
- Structure


JUMBO


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Definition of "3-partition problem"

A 3-partition problem is given by:

- A list of $3 \cdot q$ numbers.
  such as $4, 7, 9,\ 6, 3, 5,\ 2, 10, 14$
- A bound β.
  such as 20

A solution to the 3-partition problem is given by q sets of 3, where the sum of each set is β.
For example:

- 4, 2, 14
- 7, 3, 10
- 9, 6, 5

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When is 3-partition unsolvable

Knowing is solving

If we know when 3-partition is unsolvable, we can find a solution:

- Define "Candidate sets" $\mathcal{C}$ from the numbers $a_1, a_2, \ldots, a_{3q}$:

$$\mathcal{C} = \{ \ \{i, j, k\} \ \mid \ a_i + a_j + a_k = \beta\}$$

- Take $S \in \mathcal{C}$, remove those three elements and check whether the remaining numbers yield something feasible.
- Repeat until something feasible remains. Then solve the smaller instance with the remaining numbers.

$|\mathcal{C}| < q^3$, so after q^3 steps we will have obtained a smaller instance. Hence to obtain a solution, we need to perform at most q^4 checks whether 3-partition is solvable.

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When is 3-partition unsolvable

A fast approach

$$\mathcal{C} = \{ \ \{i, j, k\} \ \mid \ a_i + a_j + a_k = \beta\}$$

- Let $\mathcal{C}_i$ denote all sets in $\mathcal{C}$ containing i.
- Solve for y:

$$\sum_{C \in \mathcal{C}_i} y_C = 1 \quad \text{for all numbers } i$$

- We need $y_C \in \{0, 1\}$.
- If we allow $y_C \in [0, 1]$ there exists a fast algorithm.
- Use this algorithm, and hope it returns $y_C \in \{0, 1\}$.

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When our trick fails

- We need $y_C \in \{0, 1\}$.
- If we allow $y_C \in [0, 1]$ there exists a fast algorithm.
- In some cases, it returns $y_C \notin \{0, 1\}$.
- This means that we do not know whether a solution with $y_C \in \{0, 1\}$ exists.
- In some of these cases, the 3-partition problem is not feasible.
- Can we find such cases? (this may take some time)

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Nearly-feasible instances

- Consider an instance with these numbers:
  1,2,2, 2,3,4, 4,5,5, 7,7,7, -4,-4,-8, -10,-11,-12
  and bound $\beta = 0$.
- The problem

$$\sum_{C \in \mathcal{C}_i} y_C = 1 \quad \text{for all numbers } i$$

  has a solution.
- But it has $y_C \in \{0, 1/2\}$.
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The nearly-feasible solution

1,2,2', 2'',3,4, 4',5,5', 7,7',7'', -4,-4',-8, -10,-11,-12

$$y_{\{1,3,\text{-}4\}} = 1/2 \qquad y_{\{1,7',\text{-}8\}} = 1/2$$
$$y_{\{2,2',\text{-}4\}} = 1/2 \qquad y_{\{2,2'',\text{-}4'\}} = 1/2$$
$$y_{\{2',2'',\text{-}4'\}} = 1/2 \qquad y_{\{3,7',\text{-}10\}} = 1/2$$
$$y_{\{4,4',\text{-}8\}} = 1/2 \qquad y_{\{4,7'',\text{-}11\}} = 1/2$$
$$y_{\{4',7'',\text{-}11\}} = 1/2 \qquad y_{\{5,5',\text{-}10\}} = 1/2$$
$$y_{\{5,7,\text{-}12\}} = 1/2 \qquad y_{\{5',7,\text{-}12\}} = 1/2$$

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\begin{aligned}
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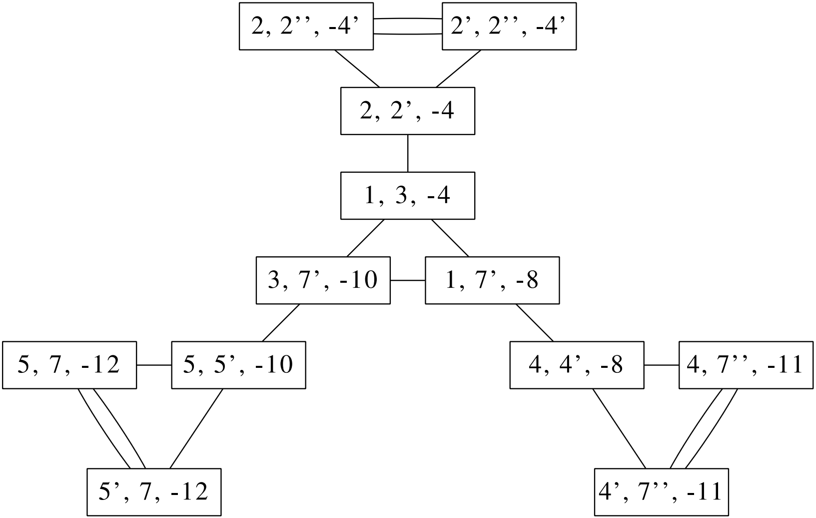
\end{aligned}
$$


2, 2’’, -4’
2’, 2’’, -4’
2, 2’, -4
1, 3, -4
3, 7’, -10
1, 7’, -8
5, 7, -12
5, 5’, -10
4, 4’, -8
4, 7’’, -11
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Using structure

- There are 14 solution-graphs with 12 vertices.
- There are 160 solution-graphs with 14 vertices.
- There are between 1050 and 1074 solution-graphs with 16 vertices.

When our trick fails

Other instance

- Instance where all solutions are $y_C \in \{0, 1/2, 1\}$ are called (nearly-)feasible.
- There is an instance with solutions $y_C \in \{0, 1/n, 2/n, ..\}$ for any $n > 1$:
  0,0,0, 1,1,2, 3,3,4, 4,4,4, 4,6,6, 9,10,11
- And one with solutions with $y_C \in \{0, 1/3, 2/3\}$:
  5,5,5, 5,5,0, 0,1,1, 2,2,4, 4,7,7, 11,13,13
- What about $y_C \in \{0, 1/4, 2/4, 3/4\}$?
- What about $y_C \in \{0, 1/5, 2/5, 3/5, 4/5\}$?

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Summary

- The "3-partition problem" occurs as part of many real problems.
- Finding "nearly-feasible" instances gives us examples that can be tough to solve.
- Such instances have a lot of structure and a relatively small size.